SYDNEY BOYS' HIGH SCHOOL

MOORE PARK, SURRY HILLS

TRIAL HSC
August 1999

MATHEMATICS

4 UNIT ADDITIONAL

Time allowed — 3 Hours
(plus 5 minutes reading time)

Examiner: E. Choy & P.S. Parker

DIRECTIONS TO CANDIDATES

- *ALL* questions may be attempted.
- All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used.
- Use a new booklet for each question.
- If required, additional booklets may be obtained from the Examination Supervisor upon request.

This is a trial paper and does not necessarily reflect the format or content of the HSC examination for this subject.

Question 1. (Start a new booklet) **Marks**

(a) Evaluate **2**

$$\int_1^3 \frac{x\,dx}{\sqrt{10-x^2}}$$

(b) (i) Evaluate **4**

$$\int \frac{dx}{x(x+1)} \text{ for } x>0$$

(ii) Hence, using the substitution $u=e^x$ evaluate

$$\int \frac{dx}{1+e^x}$$

(c) Evaluate **2**

$$\int_1^{e^2} x \ln\sqrt{x}\,dx$$

(d) Evaluate **3**

$$\int \sin^2 x \cos^5 x\,dx$$

(e) Evaluate **4**

$$\int \frac{\cos x\,dx}{\sin x+\sin^2 x}$$

Question 2. (Start a new booklet) **Marks**

(a) (i) $\dfrac{3+2i}{4-i} = a+ib$, where a and b are real. **5**

Find a and b.

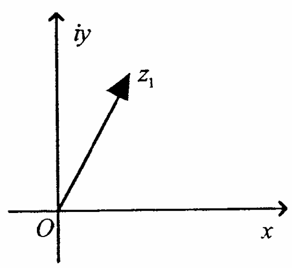
(ii) If $z = 4+3i$, express z in mod-arg form (give argument to the nearest degree) and write down the value of $\arg(z^5)$.

(b) (i) Sketch the set of points in the complex plane satisfying both inequalities. Show all intercepts with the real and imaginary axes. **6**

$|6-5i+z| \le 5$ and $\text{Im}(z) \le 5$

(ii) Sketch the locus of $\text{Re}(z\bar{z} - 2z + 8\bar{z} - 16) = 0$, showing all intercepts with the real and imaginary axes.

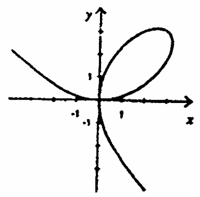
(c) **4**



The diagram above shows a vector representing the complex number z_1.
Copy the diagram and on it show vectors representing the complex numbers iz_1 and $z_1 - iz_1$.
Hence sketch a rectangle labelling the vertices in terms of z_1.

Question 3. (Start a new booklet) **Marks**

(a) The curve below is called the *folium of Descartes* and its equation is $x^3 + y^3 = 6xy$ **3**



(i) Find $\dfrac{dy}{dx}$

(ii) Find the equation of the tangent line to the folium of Descartes at the point (3, 3).

(b) Sketch $f(x) = \dfrac{x-2}{x-3}$ and hence sketch on separate number planes: **9**

(i) $y = |f(x)|$

(ii) $y = [f(x)]^2$

(iii) $y = \dfrac{1}{f(x)}$

(iv) $y = \dfrac{x-2}{|x-3|}$

(c) Sketch $(1 + x^2)y^2 = x$, making sure to indicate turning points and intercepts on your diagram. **3**

Question 4. (Start a new booklet) **Marks**

(a) (i) Use the factor theorem to factorise the polynomial **7**
$f(x) = x^4 - 4x^3 - 9x^2 + 16x + 20$

(ii) Without any use of calculus, sketch $y = x^4 - 4x^3 - 9x^2 + 16x + 20$, showing all x and y intercepts.

(iii) The polynomial $x^4 - 4x^3 - 9x^2 + (16 - m)x + 20 - b$ has two double roots α and β.

(α) Using the sum and products of roots, write down four equations involving α, β, m and b.

(β) Hence find α, β, m and b.

(iv) Hence or otherwise determine the equation of the unique line which is tangent at two distinct points to the curve $y = x^4 - 4x^3 - 9x^2 + 16x + 20$.

(b) Let α be the complex root of the equation $z^7 = 1$ with smallest positive argument. **8**
Let $\theta = \alpha + \alpha^2 + \alpha^4$ and $\phi = \alpha^3 + \alpha^5 + \alpha^6$.

(i) Explain why $\alpha^7 = 1$ and $1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 = 0$

(ii) Show that $\theta + \phi = -1$ and $\theta\phi = 2$, and hence write down a quadratic equation whose roots are θ and ϕ.

(iii) Show that $\theta = -\frac{1}{2} + \frac{i\sqrt{7}}{2}$ and $\phi = -\frac{1}{2} - \frac{i\sqrt{7}}{2}$

(iv) Write down α in mod-arg form, and show that:

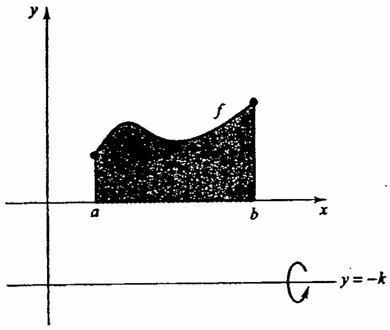
(α) $\cos\frac{4\pi}{7} + \cos\frac{2\pi}{7} - \cos\frac{\pi}{7} = -\frac{1}{2}$;

(β) $\sin\frac{4\pi}{7} + \sin\frac{2\pi}{7} - \sin\frac{\pi}{7} = \frac{\sqrt{7}}{2}$

Question 5. (Start a new booklet) **Marks**

(a) In the diagram below, $y = f(x)$ is a continuous, nonnegative function on $a \le x \le b$ **5**
and let Ω be the region between the graph of $y = f(x)$ and the x axis.
Let k be a positive constant.
Prove that if the region Ω is revolved around the line $y = -k$, then the volume of the solid of revolution is given by:

$$V = \pi\int_a^b ([f(x)+k]^2 - k^2)\, dx$$



(b) The only force acting on a particle moving in a straight line is a resistance, $mk(c+v)$, acting in that line, where m is the mass of the particle, v its velocity and k, c are positive constants. **10**
The particle had an initial velocity U (> 0) and comes to rest in time T. Its velocity is $\frac{1}{4}U$ at time $\frac{1}{2}T$.

(i) Show that $\dfrac{c+v}{c+U} = e^{-kt}$

(ii) Hence show that $c = \frac{1}{8}U$

(iii) Show also that at time t, $\dfrac{8v}{U} = 9e^{-kt} - 1$

Question 6. (Start a new booklet) **Marks**

(a) Consider the geometric series **7**

$$S = 1 + (1+x) + (1+x)^2 + \cdots + (1+x)^n$$

(i) Show that

$$S = \frac{(1+x)^{n+1} - 1}{x}$$

(ii) Hence show that

$$S = \binom{n+1}{1} + \binom{n+1}{2}x + \cdots + \binom{n+1}{r+1}x^r \cdots + \binom{n+1}{n+1}x^n$$

(iii) Hence prove

$$\binom{n}{r} + \binom{n-1}{r} + \binom{n-2}{r} + \cdots + \binom{r}{r} = \binom{n+1}{r+1}$$

(b) In how many ways can we distribute 21 books among 3 people, Martin, Barton and Hugo, so that Martin and Barton together receive twice as many books as Hugo? **4**

(c) In how many ways can the 7 letters $a\ b\ X\ X\ X\ c\ d$ be arranged in a line if no two X's are together? **4**

Question 7. (Start a new booklet) **Marks**

(a) Show that **6**

$$\int_0^1 (1-x^2)^n\,dx = \frac{2\times4\times6\times8\times\cdots\times2n}{3\times5\times7\times\cdots\times(2n+1)},\ n\geq 0$$

(b) (i) Show that $p_3(x)=1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}$ has at least one real root. **9**

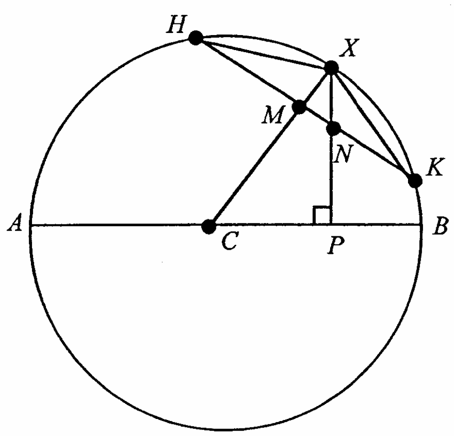
(ii) Show that $p_2(x)=1+x+\dfrac{x^2}{2!}$ has no real roots and deduce that $p_3(x)$ has exactly one real root.

(iii) Deduce that $p_4(x)=1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\dfrac{x^4}{4!}>0$ for all x.

(iv) Let $p_n(x)=\sum\limits_{k=0}^{n}\dfrac{x^k}{k!},\ n\geq 1$.
Use induction to prove that if n is even then $p_n(x)>0$ for all x and if n is odd then $p_n(x)$ has exactly one real root and this root is negative.

Question 8. (Start a new booklet) **Marks**

(a) 6



AB is a diameter of a circle, whose centre is C, and X is any point on the circumference.
P is the foot of the perpendicular drawn from X to AB.
H and K are points on the circumference such that $XH = XK$.
HK meets XP in N and XC in M.
Prove that the points M, N, C and P are concyclic.

Question 8. (continued) **Marks**

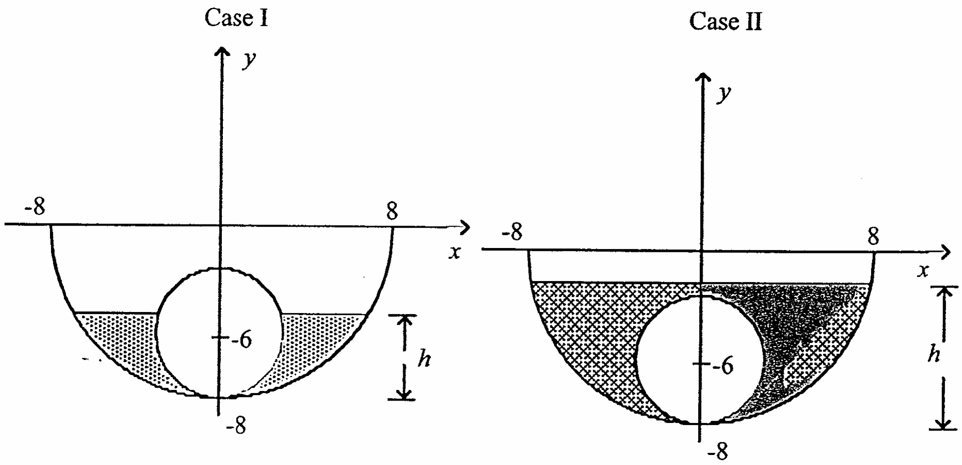
(b) A bowl shaped like a hemisphere of radius 8 cm contains a scoop of ice cream (a sphere) with a diameter of 4 cm. **9**

If the bowl is filled with milk to a depth of h cm, what is the volume of milk it contains?
You will need to consider two cases:

(I) The case where the ice cream is partially submerged.

(II) The case where the ice cream is totally submerged.

You can assume in both cases that the ice cream doesn't melt or float.



NOT TO SCALE

THIS IS THE END OF THE PAPER.

SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

1999

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Sample Solutions

Q1(a) Put $10-x^2 = u$ When $x=3$, $u=1$

$-2x\,dx = du$ ✓ $x=1$, $u=9$

So, $\displaystyle\int_1^3 \frac{x\,dx}{\sqrt{10-x^2}} = \frac{-1}{2}\int_9^1 u^{-1/2}\,du$

$= \frac{1}{2}\left[2u^{1/2}\right]_1^9$

$= 3-1$

$= 2$ ✓

(b)(i) $\displaystyle\frac{1}{x(x+1)} \equiv \frac{A}{x} + \frac{B}{x+1}$

$1 \equiv A(x+1) + Bx$

Put $x=0$, $A=1$ ✓

$x=-1$, $B=-1$

$\therefore \displaystyle\int \frac{dx}{x(x+1)} = \int x^{-1}\,dx - \int \frac{dx}{x+1}$ ✓

$= \ln\frac{x}{x+1} + C$

(ii) Put $u = e^x$

$du = e^x\,dx$

$\displaystyle\int \frac{e^x\,dx}{e^x(e^x+1)}$ ✓ $\displaystyle= \int \frac{du}{u(u+1)}$

$= \ln\frac{u}{u+1} + C$ ✓

$= x - \ln(e^x+1) + C$.

(c) $\displaystyle I = \frac{1}{2}\int_1^{e^2} x\ln x\,dx$ ✓

$\int uv' = uv - \int u'v$

$u = \ln x$ $v' = x\,dx$

$u' = \frac{dx}{x}$ $v = \frac{x^2}{2}$

$= \frac{1}{2}\left[\frac{x^2}{2}\ln x\right]_1^{e^2} - \int_1^{e^2} \frac{1}{2}\cdot\frac{x^2}{2}\cdot\frac{dx}{x}$

$= \frac{1}{4}\left(e^4.2 - 0\right) - \frac{1}{4}\left[\frac{x^2}{2}\right]_1^{e^2}$

$= \frac{e^4}{2} - \frac{1}{8}\left(e^4 - 1\right)$

$= \frac{3e^4}{8} + \frac{1}{8}$. ✓ (~ 20.60)

(d) $I = \int \sin^2 x(1-\sin^2 x)^2\cos x\,dx$ ✓

$= \int(\sin^2 x - 2\sin^4 x + \sin^6 x)\cos x\,dx$ ✓

$= \frac{\sin^3 x}{3} - \frac{2\sin^5 x}{5} + \frac{\sin^7 x}{7} + C$ ✓

Q2(a)(i) $\frac{3+2i}{4-i} \times \frac{4+i}{4+i} = \frac{12+3i+8i-2}{16+1}$

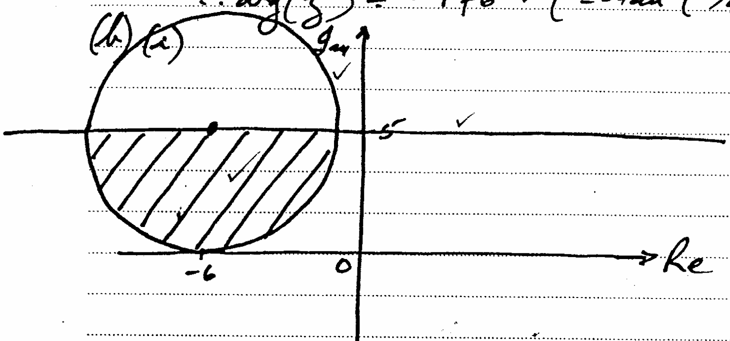
i.e. $a = \frac{10}{17}$ ✓, $b = \frac{11}{17}$ ✓

(ii) $|z| = \sqrt{16+9}$
$= 5$ ✓

$\arg z = \tan^{-1}(3/4)$ i.e. $z = 5 \operatorname{cis} 37°$
$\doteqdot 37°$ ✓

$\therefore \arg(z^5) \doteqdot -176°$ ✓ $(= 5\tan^{-1}(3/4))$

(b)(i)



(ii) Put $z = x + iy$, $z\bar{z} = x^2 + y^2$

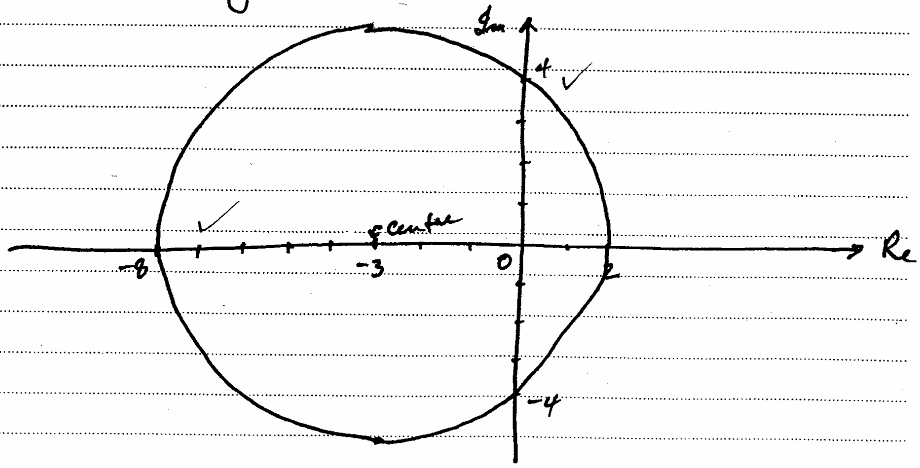
$z\bar{z} - 2z + 8\bar{z} - 16 = x^2 + y^2 - 2x - 2iy + 8x - 8iy - 6$
$= x^2 + y^2 + 6x - 16 - 10iy$

Taking the real part,

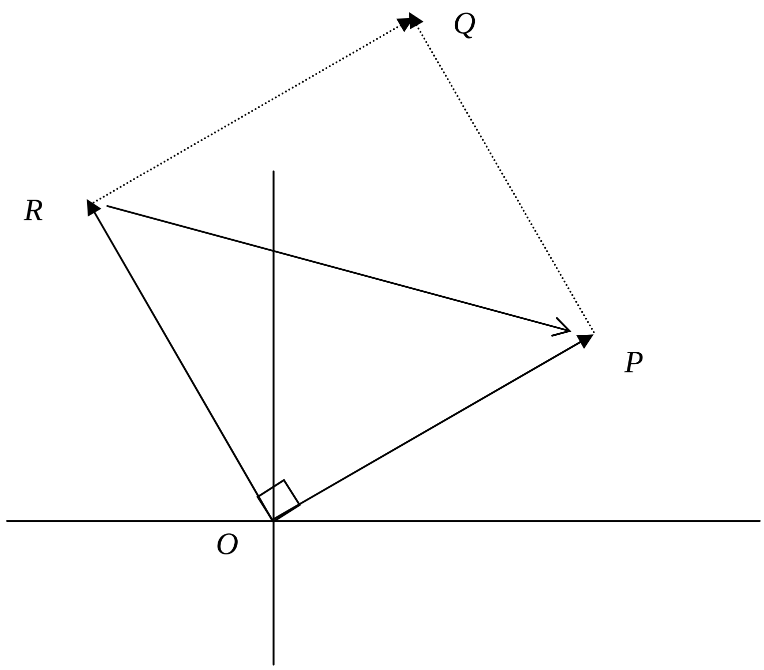
$x^2 + 6x \quad + y^2 - 16 = 0$

$x^2 + 6x + 9 \quad + y^2 = 16 + 9$

$(x+3)^2 \quad + y^2 = 5^2$ ✓



2(c)



P represents the complex number z_1, so by rotating anticlockwise by $90°$. R represents the complex number iz_1. The complex number $z_1 - iz_1$ is represented by the diagonal RP, note the direction of the vector.

So $OPQR$ is a rectangle.

Q3(a)(i) $x^3 + y^3 = 6xy$

$3x^2dx + 3y^2dy = 6ydx + 6xdy$ ✓

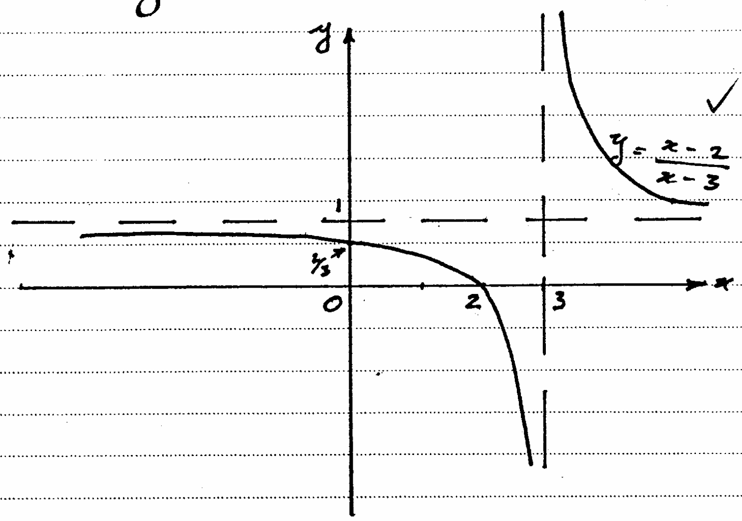
$dx(x^2 - 2y) = dy(2x - y^2)$

$\frac{dy}{dx} = \frac{x^2 - 2y}{2x - y^2}$ ✓

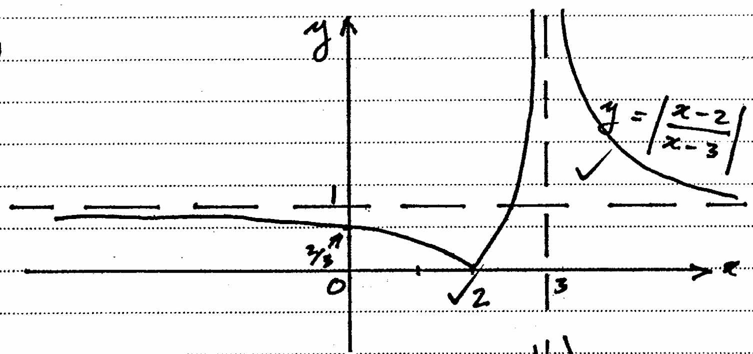
(ii) $y - 3 = \left(\frac{9-6}{6-9}\right)(x - 3)$

$x + y - 6 = 0$ ✓

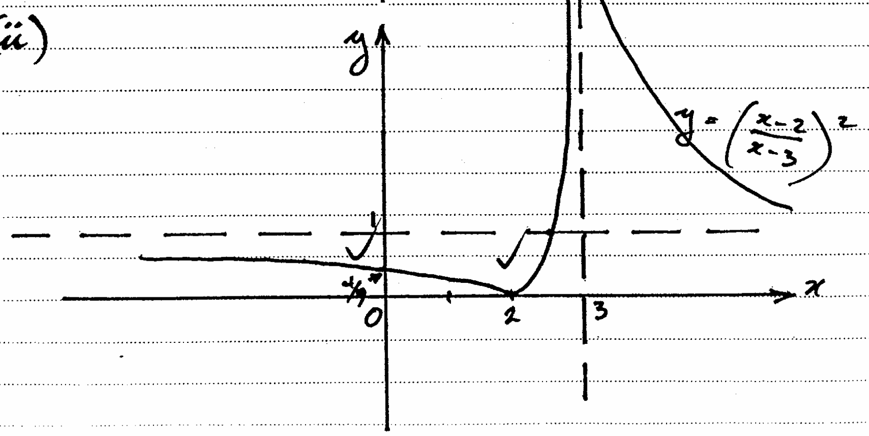
(b)



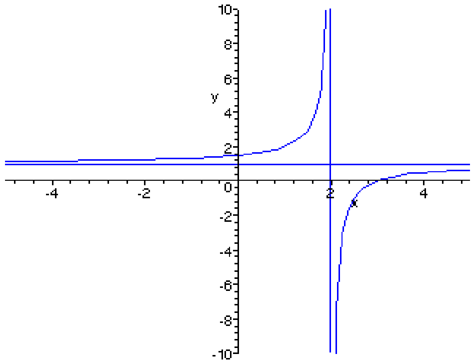
(i)



(ii)

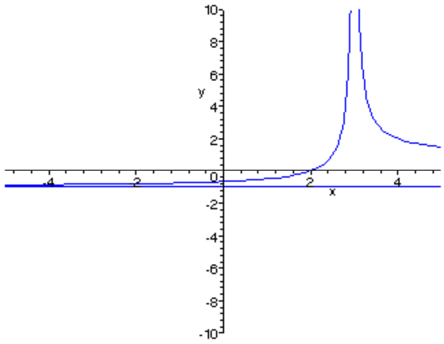


3(b) (iii) $y = 1/f(x)$



Vertical asymptote at x= 2 and a horizontal asymptote at y = 1

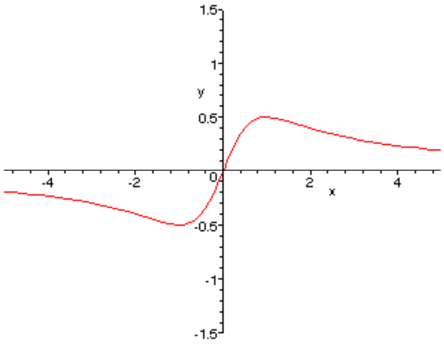
(iv)



Vertical asymptote at x= 3 and a horizontal asymptote at $y = -1$. Note that for $x < 3$ the graph is $-f(x)$.

(c) $(1+x^2)y^2 = x$

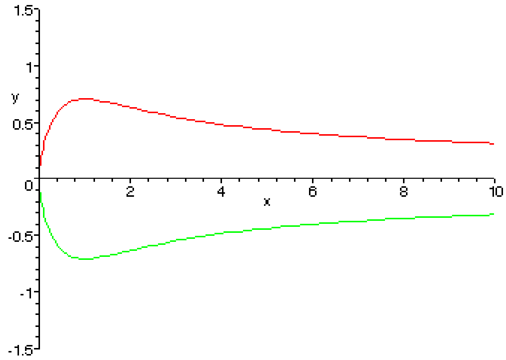
First draw $y = \dfrac{x}{(1+x^2)}$. Note that it is an odd function with turning points at $\left(\pm 1, \pm\dfrac{1}{2}\right)$



The x-axis is a horizontal asymptote.

3(c) continued

Remove the part of the graph below the x axis and reflect the positive part. At $x = 0$, the graph will have a vertical tangent.



Q4. a (i) $f(x) = x^4 - 4x^3 - 9x^2 + 16x + 20$

$f(-1) = 0$

$f(2) = 0$

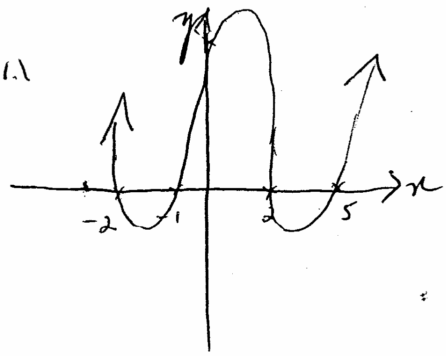
$f(5) = 0$

$f(-2) = 0$

$\boxed{f(x) = (x+1)(x-2)(x+2)(x-5)}$

1

(ii)



1

(iii) $P(x) = x^4 - 4x^3 - 9x^2 + (16-m)x + (20-b)$

Let roots be α & β

$\boxed{2\alpha + 2\beta = 4}$ or $\boxed{\alpha + \beta = 2}$ (1)

$\alpha^2 + \alpha\beta + \alpha\beta + \alpha\beta + \alpha\beta + \beta^2 = -9$

$\boxed{\alpha^2 + 4\alpha\beta + \beta^2 = -9}$ — (2)

2

$\alpha^2\beta + \alpha^2\beta + \alpha\beta^2 + \alpha\beta^2 = m - 16$

$\boxed{2(\alpha^2\beta + \alpha\beta^2) = m - 16}$ (3)

$\boxed{\alpha^2\beta^2 = 20 - b}$ (4)

From ② $(\alpha+\beta)^2 + 2\alpha\beta = -9$

$\therefore\ 2^2 + 2\alpha\beta = -9$

$2\alpha\beta = -13$

$\alpha\beta = -\frac{13}{2}$

$(\alpha\beta)^2 = \frac{169}{4} = 20-b$

$169 = 80-4b$

$89 = -4b$

$\boxed{b = -\frac{89}{4}}$

From ③ $2\alpha\beta(\alpha+\beta) = m-16$

$-13 \times 2 = m-16$

$-26 = m-16$

$\boxed{m = -10}$

B 2

also $(\alpha-\beta)^2 = (\alpha+\beta)^2 + 4\alpha\beta$

$= 4 + 26$

$= 30$

$\alpha-\beta = \pm\sqrt{30}$

$\alpha+\beta = 2$

$\therefore 2\alpha = 2 \pm \sqrt{30}$

$\left.\begin{array}{l}\alpha = 1 \pm \frac{\sqrt{30}}{2} \\ \beta = 1 \mp \frac{\sqrt{30}}{2}\end{array}\right\}$ $\boxed{\alpha \vee \beta \text{ are } 1 \pm \frac{\sqrt{30}}{2}}$

(iv) Solve $y = mx - b$

$\&\ y = x^4 - 4x^3 - 9x^2 + 16x + 20$

$\Rightarrow x^4 - 4x^3 - 9x^2 + (16-m)x + 20 - b = 0$

$\therefore$ if tangent then 2 double roots

$m = -10,\ b = -\frac{89}{4}$

$\therefore \boxed{y = -10x - \frac{89}{4}}$ or $\boxed{40x + 4y + 89 = 0}$

b, (i) $z^7 = 1$. let $z = r\,\text{cis}\,\theta$ $= \text{cis}\,\theta$. (NB $|z^7| = |z|^7 = 1$ $\therefore r = 1$)

$\text{cis}\,7\theta = 1$

$\cos 7\theta = \cos 2k\pi$

$\theta = \frac{2k\pi}{7}$ where $k \in J$.

$\therefore$ if $\alpha = \text{cis}\frac{2\pi}{7}$, $\alpha^2 = \text{cis}\frac{4\pi}{7}$ $- - -$ $\alpha^6 = \text{cis}\frac{12\pi}{7}$ are complex roots

if $k = 0$ $\theta = 0$ $\therefore 1$ is the real root.

$\therefore$ roots are $1, \alpha, \alpha^2, - - -, \alpha^6$

now $\sum \text{roots} = \frac{-b}{a} = 0$.

$\therefore$ $\boxed{1 + \alpha + \alpha^2 + \alpha^3 + \cdots + \alpha^6 = 0}$ (A) 2

& since α is a root of $z^7 = 1$

$\boxed{\alpha^7 = 1.}$

(ii) $\boxed{\theta + \phi = \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 = -1}$ from (A)

$\theta\phi = (\alpha + \alpha^2 + \alpha^4)(\alpha^3 + \alpha^5 + \alpha^6)$

$= \alpha^4(1 + \alpha + \alpha^3)(1 + \alpha^2 + \alpha^3)$

$= \alpha^4(\cancel{1 + \alpha^2 + \alpha^3} + \alpha + \alpha^3 + \alpha^4 + \alpha^3 + \alpha^5 + \alpha^6)$

$= \alpha^4(1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 \quad + \alpha^3 + \alpha^3)$

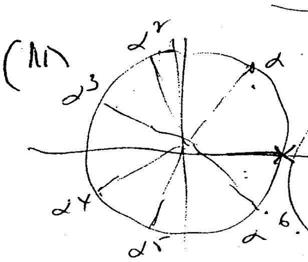
$= \alpha^4(\,0 \quad + 2\alpha^3)$

$= 2\alpha^7$ 2

$= 2.$ $\therefore$ $x^2 - S_1 x + S_2 = 0$

$\Rightarrow$ $\boxed{x^2 + x + 2 = 0}$

(iii)



From diagram $\text{Im}\,\theta > 0$ & $\text{Im}\,\phi < 0$.

$x = \frac{-1 \pm \sqrt{1-8}}{2}$

$\boxed{\theta, \phi = \frac{-1 \pm i\sqrt{7}}{2}}$ 2

$\therefore$ $\boxed{\theta = -\frac{1}{2} + \frac{i\sqrt{7}}{2}}$ & $\boxed{\phi = -\frac{1}{2} - \frac{i\sqrt{7}}{2}}$

(iv.)

now: from ⊗ $= -\frac{1}{2} + i\frac{\sqrt{7}}{2}$

$$\text{cis}\,\frac{2\pi}{7} + \text{cis}\,\frac{4\pi}{7} + \text{cis}\,\frac{8\pi}{7} = -\frac{1}{2} + i\frac{\sqrt{7}}{2}$$

equating real & imaginary parts

$$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7} = -\frac{1}{2} \quad — (1)$$

2

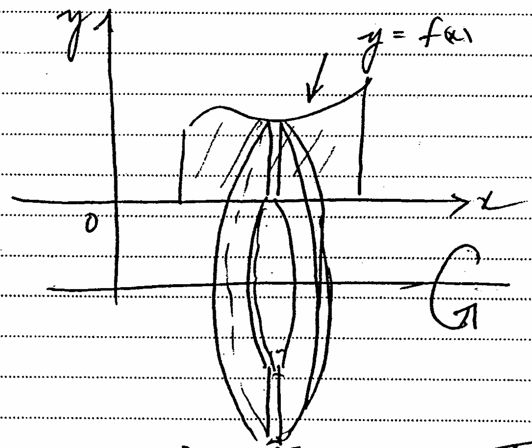
$$\& \quad \sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7} = \frac{\sqrt{7}}{2} \quad — (2)$$

from (1) $\boxed{\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} - \cos\frac{\pi}{7} = -\frac{1}{2}}$ $\left(\cos\frac{8\pi}{7} = -\cos\frac{\pi}{7}\right)$

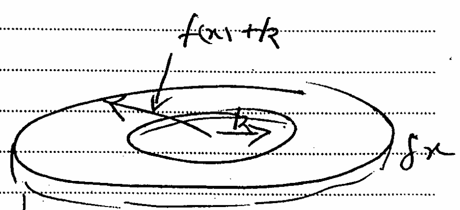
& from (2):

$\boxed{\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} - \sin\frac{\pi}{7} = \frac{\sqrt{7}}{2}}$ $\left(\sin\frac{8\pi}{7} = -\sin\frac{\pi}{7}\right)$

Q5 (a)



2 for the idea of washer technique with diagram



$$\delta V = \pi\left[[f(x)+k]^2 - k^2\right]\delta x$$

$$V = \lim_{\delta x \to 0} \sum_{x=a}^{b} \pi\left[[f(x)+k]^2 - k^2\right]\delta x \quad \downarrow 3$$

$$= \pi\int_a^b \left([f(x)+k]^2 - k^2\right)dx \quad \downarrow 5$$

5

(b) (i) $m\ddot{x} = -mk(c+v)$

$\ddot{x} = -k(c+v)$

$\frac{dv}{dt} = -k(c+v)$

$\frac{dt}{dv} = -\frac{1}{k(c+v)}$

$t = -\frac{1}{k}\ln(c+v) + d.$ when $t=0$, $v=U$

$0 = -\frac{1}{k}\ln(c+U) + d$

$d = \frac{1}{k}\ln(c+U)$

$\therefore t = -\frac{1}{k}\ln(c+v) + \frac{1}{k}\ln(c+U)$

$t = -\frac{1}{k}\ln\left(\frac{c+v}{c+U}\right)$

$-kt = \ln\left(\frac{c+v}{c+U}\right)$

$\boxed{e^{-kt} = \frac{c+v}{c+U}}$ — (A) 4

(ii) $v=0$ when $t=T$

$v=\frac{U}{4}$ when $t=\frac{T}{2}$

$\therefore$ in (A) $e^{-kT} = \frac{c+0}{c+U} \Rightarrow \boxed{e^{-kT} = \frac{c}{c+U}}$ — (B)

& $\boxed{e^{-k\frac{T}{2}} = \frac{c+\frac{U}{4}}{c+U}}$ — (C)

from (C) & (B)

$$\left(e^{-k\frac{T}{2}}\right)^2 = \left(\frac{C+\frac{U}{4}}{C+U}\right)^2 = \frac{C}{C+U}.$$

$$\left(C+\frac{U}{4}\right)^2 = C(C+U)$$

$$C^2 + \frac{CU}{2} + \frac{U^2}{16} = C^2 + CU.$$

$$\frac{U^2}{16} = \frac{1}{2}CU$$

$$\frac{U^2}{8} = CU$$

$$\therefore \boxed{C = \frac{U}{8}}$$

3

(iii) for (A)

$$e^{-kt} = \frac{\frac{U}{8}+v}{\frac{U}{8}+U}$$

$$e^{-kt} = \frac{U+8v}{U+8U}$$

$$e^{-kt} = \frac{U+8v}{9U}$$

$$9e^{-kt} = \frac{U+8v}{U}$$

$$9e^{-kt} = 1 + \frac{8v}{U}$$

$$\therefore \boxed{\frac{8v}{U} = 9e^{-kt} - 1}$$

3

Q6. $S = 1 + (1+x)^1 + (1+x)^2 + \cdots + (1+x)^n$

(i) $S = \dfrac{1\cdot\left((1+x)^{n+1} - 1\right)}{(1+x) - 1}$ [S_n of a Geo. Series NB $n+1$ terms]

$= \dfrac{(1+x)^{n+1} - 1}{x}$

2

(ii). $S = \dfrac{1 + \binom{n+1}{1}x + \binom{n+1}{2}x^2 + \cdots \binom{n+1}{r+1}x^{r+1} + \cdots \binom{n+1}{n+1}x^{n+1} - 1}{x}$

$= \binom{n+1}{1} + \binom{n+1}{2} + \cdots + \binom{n+1}{r+1}x^r + \cdots + \binom{n+1}{n+1}x^n$

2

(iii) Consider the co-eff of x^r in

$(1+x)^n = \binom{n}{r}$

$(1+x)^{n-1} = \binom{n-1}{r}$ $\therefore$ co-eff of x^r in S

$(1+x)^{n-2} = \binom{n-2}{r}$

$\vdots$

$(1+x)^r = \binom{r}{r}$ is $\binom{n}{r} + \binom{n-1}{r} + \binom{n-2}{r} + \cdots + \binom{r}{r}$

equate with co-eff of x^r in (ii)

We get

$\binom{n}{r} + \binom{n-1}{r} + \binom{n-2}{r} + \cdots + \binom{r}{r} = \binom{n+1}{r+1}$ QED

6

(b) M & B get 14 books. ie $\binom{21}{14}$ combinations

F gets the rest.

$$\therefore \binom{21}{14}\left[\binom{14}{1}+\binom{14}{2}+\cdots\binom{14}{13}\right]=\binom{21}{14}\left(2^{14}-2\right)$$

4

(c) Total ways is $\frac{7!}{3!}$

where 2 or 3 are together $= 5\times4\times4! + 5!$

$$\therefore \text{TOTAL} = \frac{7!}{3!} - \left(5\times4\times4! + 5!\right)$$

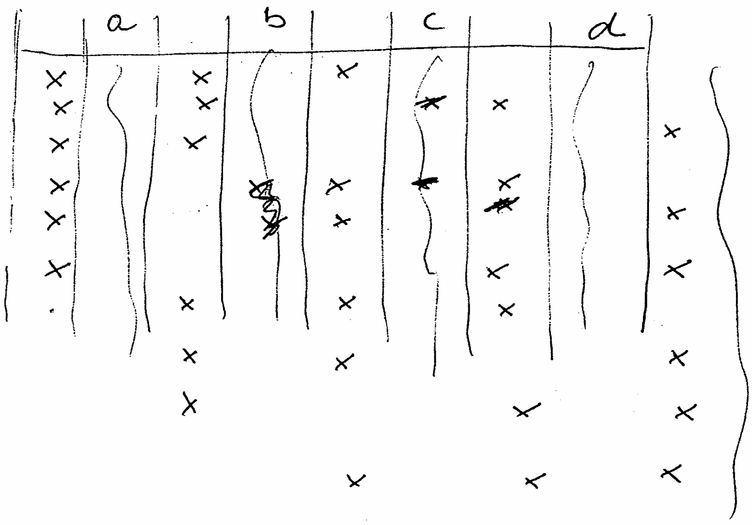
$$= 7\times5! - 5!\,(4+1)$$

$$= 2\times5!$$

$$= 240$$

4

OR.



10 ways

$\therefore$ $10\times4!$ as a, b, c, d can be arranged in $4!$ ways

$= 10\times24$

$= 240$

Question 7

(a) $I_n = \int_0^1 (1-x^2)^n dx = \int_0^1 (1-x^2)^n \frac{d(x)}{dx} dx$

$= \left[(1-x^2)^n . x\right]_0^1 - \int_0^1 x . n(1-x^2)^{n-1} . -2x\, dx$

$= 2n \int_0^1 (1-x^2)^{n-1} . x^2 dx$

$= 2n \int_0^1 (1-x^2)^n . \frac{x^2}{1-x^2} dx$

$= 2n \int_0^1 (1-x^2)^n . \left[\frac{1}{1-x^2} - 1\right] dx$

$= 2n \left[\int_0^1 (1-x^2)^{n-1} - (1-x^2)^n\right]$

$I_n = 2n\left[I_{n-1} - I_n\right]$

$\Rightarrow I_n = 2n\left[I_{n-1} - I_n\right] \quad \therefore I_n = \frac{2n}{2n+1} I_{n-1}$

Now $I_n = \frac{2n}{2n+1} . \frac{2n-2}{2n-1} . I_{n-2}$

$= \frac{2n}{2n+1} \frac{2n-2}{2n-1} \frac{2n-4}{2n-3} \cdots \frac{8}{9} \frac{6}{7} \frac{4}{5} \frac{2}{3} . 1$

$= \frac{2 \times 4 \times 6 \times \cdots \times 2n}{3 \times 5 \times 7 \times \cdots \times (2n+1)}$

(b) (i) $P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

$P_3(0) = 1$ and $P_3(x)$ is cts $\forall x$

as $P(x) \to +\infty$ as $x \to +\infty$

$P(x) \to -\infty$ as $x \to -\infty$

$\therefore$ graph of $y = P(x)$ crosses x axis at least once

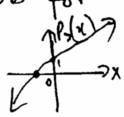
(ii) $P_2(x) = 1 + x + \frac{x^2}{2!} \quad (= P_3'(x))$

Δ of $P_2(x) = -1 < 0$

$\Rightarrow P_2(x) = 0$ has no real roots

Since there are no stat^y pts for $y = P_3(x)$ and from (i)

$\Rightarrow$ Exactly one real root.



(iii) $P_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

$P_4'(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \quad (= P_3(x))$

$\Rightarrow P_4(x)$ has one stat. point

& $P_4''(x) = 1 + x + \frac{x^2}{2} = 0$ has no real roots and since Δ of $P_4''(x)$ is < 0 and coeff. of $x^2 > 0$

$\Rightarrow P_4''(x) > 0 \quad \forall x$

ie $P_4(x)$ is concave up $\forall x$

ie MIN T.P.

and if stat. pt is at $x = \alpha$

$\Rightarrow P_4'(\alpha) = 0$

ie $P_4'(\alpha) = 1 + \alpha + \frac{\alpha^2}{2} + \frac{\alpha^3}{3!} = 0$

Now $P_4(\alpha) = 1 + \alpha + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} + \frac{\alpha^4}{4!}$

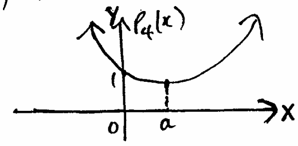
$= 0 + \frac{\alpha^4}{4!}$

> 0

$\therefore$ MIN. T.P. is above x axis

$\Rightarrow P_4(x) > 0 \quad \forall x$ since also

$y = P_4(x) \to \pm\infty$ as $x \to \pm\infty$



Possible Diagram

Q7 (iv)

$P_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$

When $n=2$, $P_2(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!}$

$\Delta < 0$, coeff of $x^2 > 0 \Rightarrow P_2(x) > 0$

$\therefore$ true for $n=2$.

Assume true for $n = 2k$ (k +ve integer)

ie assume $P_{2k}(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{2k}}{(2k)!} > 0$

Prove true for $n = 2(k+1)$

ie Prove:

$P_{2(k+1)}x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{2k}}{(2k)!} + \frac{x^{2(k+1)}}{(2k+2)!} > 0$

Now $P_{2(k+1)}x = P_{2k}(x) + \frac{\left[x^{(k+1)}\right]^2}{(2k+2)!}$

$= (+\text{ve}) + (+\text{ve})$

ie $P_{2(k+1)}x > 0 \quad \forall x$.

$\therefore$ Whenever $P_{2k}(x) > 0$ then $P_{2(k+1)}x > 0 \Rightarrow P_n(x)$ is true for $n = 2, 4, 6, \ldots$

If $n=1$, $P_1(x) = 1 + x = 0$ when $x = -1$

$\therefore P_1(x)=0$ has one real root.

Assume true for $n = 2k+1$ (k +ve integer).

& that $P_{2k+1}(\alpha) = 0$ where $\alpha < 0$.

ie Assume

$P_{2k+1}(\alpha) = 1 + \alpha + \frac{\alpha^2}{2!} + \cdots + \frac{\alpha^{2k+1}}{(2k+1)!} = 0$

RTP true for $n = 2(k+1)+1$

ie for $n = 2k+3$

Now

$P_{2k+3}(\alpha) = 1 + \alpha + \frac{\alpha^2}{2!} + \cdots + \frac{\alpha^{2k+1}}{(2k+1)!} + \frac{\alpha^{2k+2}}{(2k+2)!} + \frac{\alpha^{2k+3}}{(2k+3)!}$

$= 0 + \frac{\alpha^{2k+3}}{(2k+3)!}$

$= 0 + \frac{\alpha^{2k+2}}{(2k+2)!} + \frac{\alpha^{2k+3}}{(2k+3)!}$

$= \frac{(2k+3)\alpha^{2k+2} + \alpha^{2k+3}}{(2k+3)!}$

$= 0$

when $\alpha^{2k+2}\left[2k+3+\alpha\right] = 0$

ie when $\alpha = -2k-3 < 0$

since k is positive.

Question 8(a)

Equal chords HX and KX subtend equal angles at the centre of the circle $\Rightarrow H\hat{C}M = K\hat{C}M$

Now in ΔHCM and ΔKCM

$H\hat{C}M = K\hat{C}M$ (reason given above)

$HC = KC$ (radii of same circle)

CM is common

$\therefore \Delta HCM \equiv \Delta KCM$ by SAS test

$\Rightarrow$ the corresponding angles of the triangles are equal

ie, $H\hat{M}C = K\hat{M}C$

and since they are adjacent angles on a straight line

$\Rightarrow H\hat{M}C + K\hat{M}C = 180^\circ$

ie $H\hat{M}C = K\hat{M}C = 90^\circ$

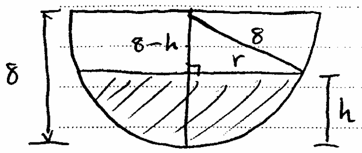
Now in quadrilateral CMNP the opposite angles CMN and CPN are supplementary

$\Rightarrow M\hat{C}P$ and $M\hat{N}P$ are supplementary (angle sum of quad^l 360°)

$\Rightarrow$ CMNP is a cyclic quadrilateral

$\therefore$ M, N, C, P are concyclic.

(b) Case (I): Without Ice Cream, the volume of milk [redacted] can be found. It has surface radius



$$r = \sqrt{8^2 - (8-h)^2}$$
$$= \sqrt{64 - 64 + 16h - h^2}$$
$$= \sqrt{16h - h^2}$$

$\therefore$ Suface Area $= \pi r^2 = \pi(16h - h^2)$

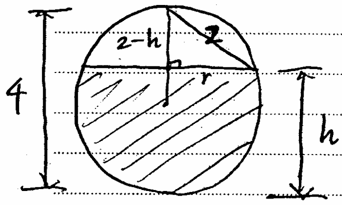
A thin slice of the milk has volume

$$\delta V = \pi(16h - h^2)\,\delta h.$$

$\therefore$ V of milk a depth of h cm has volume

$$V = \lim_{\delta h \to 0} \sum_{h=0}^{h} \pi(16h - h^2)\,\delta h$$
$$= \pi \int_0^h (16h - h^2)\,dh$$
$$= \pi\left[8h^2 - \frac{h^3}{3}\right]_0^h$$
$$= \pi\left(8h^2 - \frac{h^3}{3}\right) u^3.$$

Now Consider partically submerged Ice cream volume:



$$r = \sqrt{4 - (2-h)^2}$$
$$= \sqrt{4h - h^2}.$$

$\therefore$ Area $= \pi(4h - h^2)$

$\therefore$ $\delta V = \pi(4h - h^2)\,\delta h.$

$$V = \pi \int_0^h (4h - h^2)\,dh$$
$$= \pi\left[2h^2 - \frac{h^3}{3}\right]_0^h$$
$$= \pi\left(2h^2 - \frac{h^3}{3}\right)$$

$\therefore$ For Case (I) Volume of milk

$$= \pi\left(8h^2 - \frac{h^3}{3} - 2h^2 + \frac{h^3}{3}\right) = 6\pi h^2\ u^3.$$

For Case (II) Volume of Ice cream $= \frac{4}{3}\pi r^3$

$$= \frac{4}{3}\pi \times 8$$

$$= \frac{32}{3}\pi.$$

$\therefore$ Volume of milk $= \pi\left(8h^2 - \frac{h^3}{3} - \frac{32}{3}\right)$

$$= \frac{\pi(24h^2 - h^3 - 32)}{3} \ u^3$$